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Note

A Pseudo-Upstream Differencing Scheme for Advection

INTRODUCTION

The accurate and efficient representation of the advective process is often of major importance in the numerical modelling of fluid dynamical problems. For example, in numerical weather prediction it is of paramount interest to obtain an accurate estimate of the rate of propogation of major low-pressure systems and surface fronts. An appreciation of the behavior of the various approximation techniques that have been employed to model this process can be obtained from an analysis of their treatment of wave solutions of the prototype scalar advection equation:

$$\frac{\partial u}{\partial t} + c\frac{\partial u}{\partial x} = 0, \tag{1}$$

where c is a constant advection velocity.

A comparison of the properties of several well-established finite-difference approximations of this equation formulated within an Eulerian framework was presented by Morton [8]. It was shown that in general the magnitude of the numerical damping and/or the false dispersion due to numerical phase errors is large for conventional low-order schemes (e.g., upstream differencing) and that it was thus desirable to have recourse to more complicated, but not necessarily higher-order, schemes. Examples of such schemes are those of Roberts and Weiss [11] and Fromm [4]. These conclusions have been reinforced in subsequent studies [5, 7, 12].

A further repercussion of the desirability of adopting more complicated differencing schemes arises from the fact that these schemes also involve a considerable increase in computation. Thus other numerical approaches become viable alternatives, viz., spectral methods [7, 10], finite-element methods [3, 9], and other variants [1, 6, 13]. These alternative techniques have been shown to possess comparatively small numerical phase speed and damping errors, and concomitantly may also satisfy, subject to time-truncation errors, some of the integral properties satisfied by the original differential equations. It is to be noted, however, that some of these methods are less attractive if the boundaries of the flow domain are open, i.e., advection can take place across the "purely geometric" boundaries.

In this note we introduce a comparatively simple finite-difference scheme which is particularly suitable for use with open-boundary flow problems. The scheme is shown to possess phase and damping properties that are on a par with sophisticated difference schemes [4, 5, 11] and the alternative methods referred to earlier.

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Pseudo-Upstream Schemes

The schemes to be considered are a generalisation of the conventional upstream differencing scheme (i.e., forward in time, backward in space). First we note that the characteristics of Eq. (1) are given by the family of straight lines $x - ct = \text{const.}$ Thus, if we consider the space–time mesh given by $x = j(\Delta x)$, $t = n(\Delta t)$ with $j = 1, 2,..., n = 1, 2,...$, it follows that $u_j^{n+1} = u_c^n$, where $x_c = j(\Delta x) - c(\Delta t) = \Delta x(j - \alpha)$, and $\alpha = c(\Delta t/\Delta x)$ is the Courant number.

Finite-difference schemes can be constructed by employing spatial interpolation schemes to estimate u_c^n (and hence u_j^{n+1}) from the known grid point values of u at time level n. The conventional upstream scheme,

$$u_j^{n+1} = (1 - \alpha)\, u_j^n + \alpha u_{j-1}^n,$$

clearly corresponds to a linear interpolation for u_c^n between the two neighboring points $(j - 1, j)$. Similarly it can be shown that the Lax–Wendroff scheme is equivalent to a quadratic interpolation procedure using the grid points $(j - 1, j, j + 1)$. This strategy invites extension to higher order. The formulas derived using cubic or quintic polynomial interpolation centered about the points $(j - 1, j)$ may be termed "pseudo-upstream schemes" since they require an "upstream" strategy for their execution. In addition they also retain the original upstream scheme's property of having zero phase errors for $\alpha = 0.5$ and 1.0.

The change effected by increasing the order of the polynomial interpolation is illustrated in Fig. 1 which shows the damping and relative phase change per time step for wave solutions of Eq. (1), for the various orders of interpolation, and with $\alpha = 0.2$ and 0.7. We note that the cubic scheme is superior to the "quartic" scheme, viz., the scheme based upon a fourth-order polynomial interpolation centered about (j), in terms of phase speed and is only marginally inferior in its amplitude response. Moreover, there is a suggestion that the solutions for the odd-order polynomial schemes are rapidly asymptoting toward the correct behavior.

The difference formulation for the odd-order schemes, obtained using Lagrange's interpolation formula, takes the forms

$$\textit{Cubic:}\quad u_j^{n+1} = \tfrac{1}{2}(\delta_1\delta_2\delta_3)\{u_j^n\} - \tfrac{1}{6}\alpha\delta_1\{\delta_3 u_{j+1}^n + \delta_2 u_{j-2}^n\} + \tfrac{1}{2}\alpha\delta_2\delta_3\{u_{j-1}^n\}; \qquad (2)$$

$$\textit{Quintic:}\quad u_j^{n+1} = \tfrac{1}{12}(\delta_1\delta_2\delta_3\delta_4\delta_5)\{u_j^n\} - \mu\delta_3\delta_4\delta_5\{\tfrac{1}{2}\delta_1 u_{j+1}^n - \delta_2 u_{j-1}^n\}$$
$$+ \tfrac{1}{2}\mu\delta_1\delta_2\delta_5\{\tfrac{1}{5}\delta_3 u_{j+2}^n - \delta_4 u_{j-2}^n\} + \tfrac{1}{10}\mu\delta_1\delta_2\delta_3\delta_4\{u_{j-3}^n\}; \qquad (3)$$

$$\mu = \tfrac{1}{12}\alpha,$$
$$\delta_{1,2} = (1 \mp \alpha),$$
$$\delta_{3,4} = (2 \mp \alpha),$$
$$\delta_5 = (3 - \alpha).$$

The amplitude response and phase defects of these schemes representation of wave

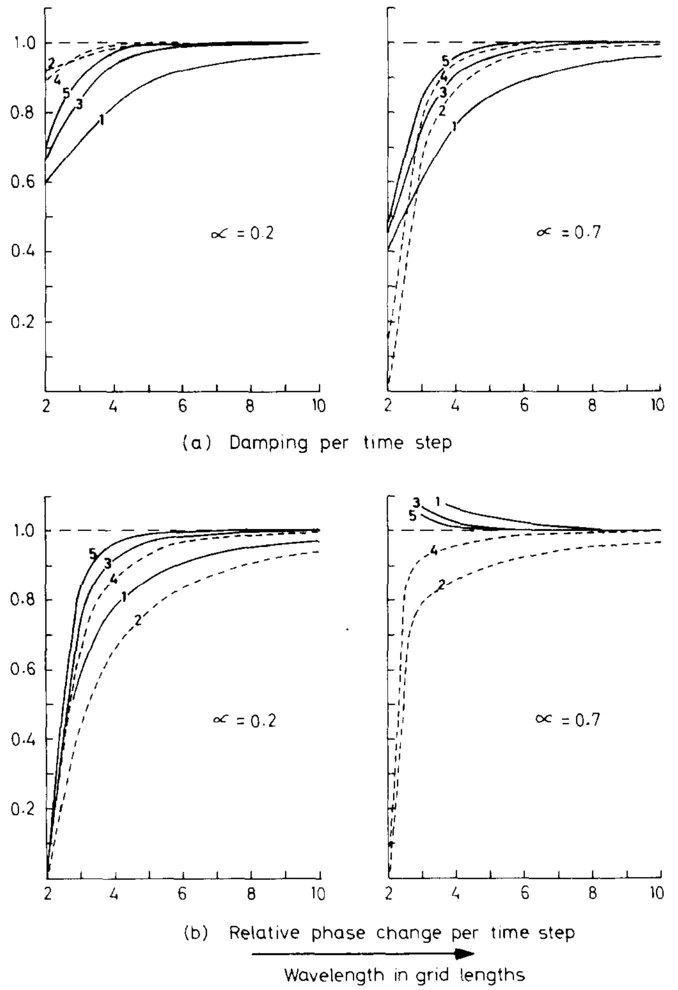

FIG. 1. (a) Damping per time step and (b) relative phase change per time step plotted as a function of the wavelength (in grid lengths) for both $\alpha = 0.2$ and $\alpha = 0.7$. The numerals on the curves refer respectively to the order of the interpolation scheme (e.g., 5 denotes the quintic scheme).

solutions of Eq. (1) are readily evaluated and the numerical values derived for the quintic are listed in Tables I and II. It is seen that, for the values displayed, the damping and relative phase speed change per time step is correct to within 5.3% at four grid lengths and correct to within 0.6% at six grid lengths and longer. The corresponding values for the cubic scheme are respectively 12% at four grid lengths and 2.7% for six grid lengths and longer wavelengths.

TABLE I

Damping per Time Step as a Function of the Courant Number (α) and the Wavelength in Grid Lengths (n) of the "Quintic" Pseudo-Upstream Scheme

n \ α	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9	1.0
2	0.868	0.692	0.481	0.246	0.000	0.246	0.481	0.692	0.868	1.000
3	0.950	0.894	0.843	0.806	0.793	0.806	0.843	0.894	0.950	1.000
4	0.986	0.972	0.961	0.953	0.950	0.953	0.961	0.972	0.986	1.000
5	0.996	0.991	0.988	0.986	0.985	0.986	0.988	0.991	0.996	1.000
6	0.998	0.997	0.996	0.995	0.995	0.995	0.996	0.997	0.998	1.000
8	1.000	0.999	0.999	0.999	0.999	0.999	0.999	0.999	1.000	1.000
10	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000

TABLE II

Relative Phase Change per Time Step as a Function of the Courant Number (α) and the Wavelength in Grid Lengths (n) of the "Quintic" Pseudo-Upstream Scheme

n \ α	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9	1.0
2	0.000	0.000	0.000	0.000	1.000	0.000	0.000	0.000	0.000	1.000
3	0.783	0.833	0.891	0.949	1.000	1.034	1.047	0.833	0.643	1.000
4	0.947	0.962	0.976	0.990	1.000	1.007	1.010	1.010	1.006	1.000
5	0.984	0.989	0.993	0.997	1.000	1.002	1.003	1.003	1.002	1.000
6	0.994	0.996	0.997	0.999	1.000	1.001	1.001	1.001	1.001	1.000
8	0.999	0.999	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000
10	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000

Further Remarks

In the previous section we merely analyzed and catalogued the phase speed and numerical damping properties of the pseudo-upstream schemes. However, the selection of a particular numerical scheme to study a specified flow problem will clearly depend upon the relative merits of that scheme for the problem under consideration. Thus, it is appropriate to provide a brief comparative assessment of the pseudo-upstream schemes.

In Fig. 2 the damping and relative phase change per time step of the cubic and quintic schemes are compared with those of the schemes recently considered by Gadd [5] and Mahrer and Pielke [6] for the case of $\alpha = 0.2$. A quantitative comparison for a range of α can be undertaken by reference to the tables in the original papers. Again a comparison of the phase change properties with that of the Fromm scheme [4] and the Roberts and Weiss scheme [11] is readily inferred from the work of

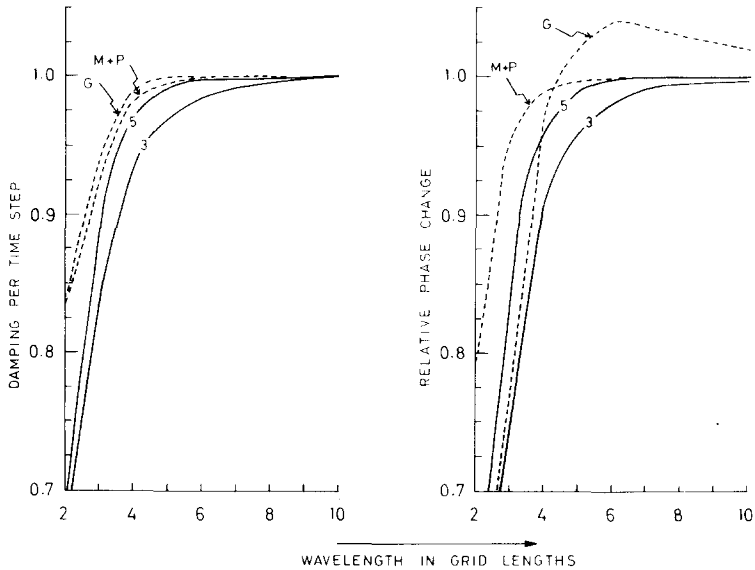

FIG. 2. (a) Damping per time step and (b) relative phase change per time step plotted as a function of the wavelength (in grid lengths) with $\alpha = 0.2$, for the cubic and quintic pseudo-upstream schemes and for the schemes of Gadd (G), and Mahrer and Pielke ($M + P$).

Morton [8]. In general, we conclude that the performance of both the cubic and quintic schemes is comparable with that of these other sophisticated difference schemes. It is also worth noting that the numerical damping of the upstream schemes is probably much less in many flow problems than that attributable to other physical diffusive mechanisms.

For the case of the linear advection equation the response of a scheme to an arbitrary initial distribution of the advected variable can clearly be described, albeit perhaps cumbersomely, in terms of the phase and damping properties, e.g., the accuracy of the representation of the group velocity of a wave packet will be related intimately to the magnitude of the phase defect. However, other more physically orientated "local" criteria might also be of crucial interest and importance in particular problems. For example, many schemes exhibit an apparent predilection to develop spurious wave-like features or even negative values for a physically positive-definite flow variable when the advected variable has a sharp gradient on the scale of the grid length. To combat these errors special schemes have been developed to guarantee "monotonicity" [14] or with "positivity-preserving" features [2]. The pseudo-upstream method, unlike its progenitor—the upstream scheme, does not formally satisfy these latter criteria. Nevertheless, a measure of the pseudo-upstream scheme's behavior in this context can be derived as follows: consider the response to the pathological instantaneos distribution of the advected variable given by an isolated spike with an amplitude ε embedded in an otherwise uniform distribution of unit amplitude. Then for $0 < \alpha < 1$ no negative values of the advected variable will be generated at the subsequent time level by the cubic and quintic schemes respectively if the amplitude of the spike is such that $\varepsilon < 16$ and $\varepsilon < 11$. We note that for the Lax–Wendroff scheme and the second-order centered leapfrog scheme the corresponding amplitude bounds are respectively $\varepsilon < 3$ and $\varepsilon < 2$.

Other criteria that influence the adoption of a scheme for a particular problem are the ease with which boundary conditions, including open-boundary effects, can be accommodated in the program and the computational efficiency of the scheme. The implementation of open-boundary conditions does not present as formidable a problem for the polynomial interpolation schemes compared with the nature of the problem for, say, the spectral technique or the spline scheme. In using the interpolation schemes with open boundaries it would appear natural to reduce the order of interpolation in the vicinity of outflow boundaries. For example, with the quintic scheme we would need to reduce the order to a cubic for grid points adjacent to the boundary and then use the scheme based upon a linear interpolation procedure at the boundary itself.

In terms of efficiency Eqs. (2) and (3) indicate that these schemes require less storage, and comparable (in the case of the cubic scheme) or considerably more (in the case of the quintic scheme) mathematical operations than, say, a difference scheme that is leapfrog in time and fourth order in space. The extension of the cubic scheme to two space dimensions would appear to be straightforward using the strategy of interpolation in two dimensions. However, for computational economy, it might be necessary to adopt the "splitting method" when extending the quintic scheme to two dimensions and acknowledge that this might entail an additional loss in accuracy.

In summary the pseudo-upstream difference schemes outlined in this note are seen to possess several attractive properties. In particular they appear to be worthy contenders for adoption in modeling fluid-flow problems that involve open boundaries and require an accurate representation of the advective process.

References

1. R. Bellman, B. G. Kashef, and J. Casti, *J. Comput. Phys.* **10** (1972), 40–52.
2. J. P. Boris and D. L. Book, *J. Comput. Phys.* **20** (1976), 397–431.
3. M. J. P. Cullen, *Quart. J. Roy. Met. Soc.* **100** (1974), 555–562.
4. J. E. Fromm, *J. Comput. Phys.* **3** (1968), 176–189.
5. A. J. Gadd, *Quart. J. Roy. Met. Soc.* **104** (1978), 583–594.
6. Y. Mahrer and R. A. Pielke, *Mon. Wea. Rev.* **106** (1978), 818–830; *Mon. Wea. Rev.* **106** (1978), 1758.
7. H.-O. Kreiss and J. Oliger, *Tellus* **24** (1972), 199–215.
8. K. W. Morton, *Proc. Roy. Soc. London Ser. A* **323** (1971), 237–253.
9. J. T. Oden, O. C. Zienkiewicz, R. H. Gallagher, and C. Taylor, "Finite Element Methods in Flow Problems," U.A.H. Press , 1974.
10. S. A. Orsag, *J. Fluid Mech.* **49** (1971), 75–113.
11. K. V. Roberts and N. O. Weiss, *Math. Comp.* **20** (1966), 272–299.

12. R. SHAPIRO, "Comparison of Strategies in Numerical Integration of a Non-linear Advection Equation," Air Force Cambridge Research Laboratories, Environmental Research Paper No. 504 (1975) Report No. AFCRL-TR-75-0212.
13. J. STEPPELER, *J. Comput. Phys.* **19** (1975), 390–403.
14. B. VAN LEER, *J. Comput. Phys.* **32** (1979), 101–136.

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HUW C. DAVIES

Department of Meteorology
University of Reading
Reading, England